

On the approximate jacobian Newton diagrams of an irreducible plane curve

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Abstract

We introduce the notion of an approximate jacobian Newton diagram which is the jacobian Newton diagram of the morphism $(f^{(k)}, f)$, where f is a branch and $f^{(k)}$ is a characteristic approximate root of f . We prove that the set of all approximate jacobian Newton diagrams is a complete topological invariant. This generalizes theorems of Merle and Ephraïm about the decomposition of the polar curve of a branch.

1 Introduction

Every two complex series $f, g \in \mathbf{C}\{x, y\}$ such that $f(0, 0) = g(0, 0) = 0$ define a germ of a holomorphic mapping $(g, f) : (\mathbf{C}^2, 0) \rightarrow (\mathbf{C}^2, 0)$. Assume that the curves $f = 0$ and $g = 0$ share no common component. Then the critical locus of this mapping is a germ of an analytic curve and its direct image by (g, f) is also an analytic curve called the *discriminant curve*. Let $D(u, v) = 0$ be an equation of the discriminant curve in the coordinates $(u, v) = (g(x, y), f(x, y))$. We call the Newton diagram of $D(u, v)$ the *jacobian Newton diagram* of the morphism (g, f) and denote it $\mathcal{N}_J(g, f)$.

Note that if $g = 0$ is a smooth curve transverse to $f = 0$ then $\mathcal{N}_J(g, f)$ is the jacobian Newton diagram of the curve $f = 0$ introduced in [Te3]. With these assumptions Teissier proves in [Te1] that $\mathcal{N}_J(g, f)$ depends only on the topological type of the curve $f = 0$.

Merle in [Me] studies the case of a smooth curve $g = 0$ transverse to an irreducible singular curve $f = 0$. He gives a description of the jacobian Newton diagram in terms of other invariants of singularity of a curve $f = 0$. He also shows that the datum of the jacobian Newton diagram determines the equisingularity class of the curve (or equivalently its embedded topological type). Ephraïm in [Eph] extends Merle's result to any smooth curve $g = 0$.

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Let f be an irreducible Weierstrass polynomial. In this paper we generalize the results of Merle to the family $\{\mathcal{N}_J(f^{(k)}, f)\}_k$, where $f^{(k)}$ is the k -th characteristic approximate root of f introduced in [A-M]. We prove, in two different ways, that this family is a complete topological invariant of the branch $f = 0$. Our computations are based on the decomposition of the critical locus of the mapping $(f^{(k)}, f)$, which is analogous to the decomposition of the polar curve obtained by Merle in [Me].

2 Plane branches, semigroup and approximate roots

We mean by the fractional power series the elements of the ring $\mathbf{C}\{x\}^* = \bigcup_{n \in \mathbf{N}} \mathbf{C}\{x^{1/n}\}$. For every two fractional power series δ and δ' we call the number $\mathcal{O}(\delta, \delta') = \text{ord}_x(\delta(x) - \delta'(x))$ the *contact order* between δ and δ' .

Every convergent power series $g(x, y) \in \mathbf{C}\{x, y\}$, $g(0, 0) = 0$ has a Newton-Puiseux factorization

$$g(x, y) = u(x, y)x^N \prod_{i=1}^d (y - \gamma_i(x)),$$

where $u(x, y) \in \mathbf{C}\{x, y\}$, $u(0, 0) \neq 0$, N is a nonnegative integer and $\gamma_i(x)$ are fractional power series of positive order. We will call γ_i the Newton-Puiseux roots of g and denote the set $\{\gamma_1, \dots, \gamma_d\}$ by Zerg .

Let $f(x, y)$ be an irreducible power series such that $\text{ord}_y(f(0, y)) = n \geq 1$. Then f has a Newton-Puiseux root of the form $\gamma_1(x) = \sum_{i=1}^{\infty} a_i x^{i/n}$. The other Newton-Puiseux roots are $\gamma_j(x) = \sum_{i=1}^{\infty} a_i \omega^{(j-1)i} x^{i/n}$ for $1 \leq j \leq n$, where $\omega \in \mathbf{C}$ is an n -th primitive root of unity. The contact orders between the elements of $\text{Zer} f$ form a set $\{b_1/n, \dots, b_g/n\}$, where $b_1 < b_2 < \dots < b_g$ and $\gcd(n, b_1, \dots, b_g) = 1$. We put $b_0 = n$ and call the sequence (b_0, b_1, \dots, b_g) the *Puiseux characteristic* of f . By convention $b_{g+1} = +\infty$.

Let A and B be finite sets of fractional power series. The *contact* $\text{cont}(A, B)$ is by definition $\max\{\mathcal{O}(\alpha, \beta) : \alpha \in A, \beta \in B\}$. If $\alpha(x)$ is a fractional power series and $f(x, y)$, $g(x, y)$ are irreducible power series co-prime to x then by abuse of notation we will write $\text{cont}(\alpha, f) := \text{cont}(\{\alpha\}, \text{Zer} f)$ and $\text{cont}(f, g) := \text{cont}(\text{Zer} f, \text{Zer} g)$.

It is well-known (see for example Lemma 4.3 of [Ca1]) that for every Newton-Puiseux root α of f we have $\text{cont}(\alpha, g) = \text{cont}(f, g)$. The contact between irreducible power series has a strong triangle inequality property: if $h_i \in \mathbf{C}\{x, y\}$ for $i = 1, 2, 3$ are irreducible power series co-prime to x then $\text{cont}(h_1, h_2) \geq \min(\text{cont}(h_1, h_3), \text{cont}(h_2, h_3))$.

In [A-M] the authors introduce the concept of *approximate root* as a consequence of the following proposition:

Proposition 1 *Let \mathbf{A} be an integral domain. If $f(y) \in \mathbf{A}[y]$ is monic of degree d and p is invertible in \mathbf{A} and divides d , then there exists a unique monic polynomial $g(y) \in \mathbf{A}[y]$ such that the degree of $f - g^p$ is less than $d - \frac{d}{p}$.*

This allows us to define:

Definition 1 *The unique monic polynomial of the preceding proposition is called the p -th approximate root of f .*

Let $f \in \mathbf{C}\{x\}[y]$ be an irreducible Weierstrass polynomial with Puiseux characteristic (b_0, \dots, b_g) . Put $l_k := \gcd(b_0, \dots, b_k)$. In particular l_k divides $\deg f = b_0$ for all $k \in \{0, \dots, g\}$. In the sequel for $k \in \{0, \dots, g-1\}$ we denote $f^{(k)}$ the l_k -th approximate root of f and we call these polynomials the *characteristic approximate roots* of f . By convention we put $f^{(-1)} = x$.

The following proposition is the main one in [A-M] (see also [G-Pl2] and [Po]):

Proposition 2 *Let $f \in \mathbf{C}\{x\}[y]$ be an irreducible Weierstrass polynomial with Puiseux characteristic (b_0, \dots, b_g) . Then the characteristic approximate roots $f^{(k)}$ for $k \in \{0, \dots, g-1\}$, have the following properties:*

1. *The polynomial $f^{(k)}$ is irreducible with Puiseux characteristic $(b_0/l_k, \dots, b_k/l_k)$.*
2. *The y -degree of $f^{(k)}$ is equal to b_0/l_k and $\text{cont}(f, f^{(k)}) = b_{k+1}/b_0$.*

Example 1 *Take the irreducible Weierstrass polynomial $f = (y^3 - 6x^3y - x^4)^2 - 9x^9$ of Puiseux characteristic $(6, 8, 11)$. The characteristic approximate roots of f are $f^{(0)} = y$ and $f^{(1)} = y^3 - 6x^3y - x^4$. The Newton-Puiseux roots of f are of the form $y = \omega^8 x^{4/3} + 2\omega^{10} x^{5/3} + \omega^{11} x^{11/6} + \dots$, where $\omega^6 = 1$ while the Newton-Puiseux roots of $f^{(1)}$ are $y = \epsilon^4 x^{4/3} + 2\epsilon^5 x^{5/3} - \frac{8}{3}x^2 + \dots$, where $\epsilon^3 = 1$. One can check directly that $\text{cont}(f, f^{(0)}) = 8/6$ and $\text{cont}(f, f^{(1)}) = 11/6$.*

3 Jacobian Newton diagrams

In this section we recall the notion of the jacobian Newton diagrams and we establish some preliminary results which are necessary for the next.

Write $\mathbf{R}_+ = \{x \in \mathbf{R} : x \geq 0\}$. Let $f \in \mathbf{C}\{x, y\}$, $f(x, y) = \sum a_{i,j} x^i y^j$ be a non-zero convergent power series. Put $\text{supp } f := \{(i, j) : a_{i,j} \neq 0\}$ the *support* of f . By definition the *Newton diagram* of f in the coordinates (x, y) is

$$\Delta_f := \text{Convex Hull}(\text{supp } f + \mathbf{R}_+^2).$$

An important property of Newton diagrams is that the Newton diagram of a product is the Minkowski sum of Newton diagrams. One has $\Delta_{fg} = \Delta_f + \Delta_g$, where $\Delta_f + \Delta_g = \{a + b : a \in \Delta_f, b \in \Delta_g\}$. In particular if f and g differ by an invertible factor $u \in \mathbf{C}\{x, y\}$, $u(0, 0) \neq 0$ then $\Delta_f = \Delta_g$. Thus the

Newton diagram of a plane analytic curve is well defined because an equation of an analytic curve is given up to invertible factor, where an analytic plane curve is a principal ideal of the ring of convergent power series $\mathbf{C}\{x, y\}$, which we will denote by $f(x, y) = 0$. We will write $\Delta_{f=0}$ for the Newton diagram of the curve $f = 0$.

Following Teissier [Te2] we introduce *elementary Newton diagrams*. For $m, n > 0$ we put $\{\frac{n}{m}\} = \Delta_{x^n + y^m}$. We put also $\{\frac{n}{\infty}\} = \Delta_{x^n}$ and $\{\frac{\infty}{m}\} = \Delta_{y^m}$.

Every Newton diagram $\Delta \subsetneq \mathbf{R}_+^2$ has a unique representation $\Delta = \sum_{i=1}^r \{\frac{L_i}{M_i}\}$, where *inclinations* of successive elementary diagrams form an increasing sequence (by definition the inclination of $\{\frac{L}{M}\}$ is L/M with the conventions that $L/\infty = 0$ and $\infty/M = +\infty$). We shall call this representation the *canonical decomposition* of Δ .

Let $\sigma = (g, f) : (\mathbf{C}^2, 0) \rightarrow (\mathbf{C}^2, 0)$ be an analytic mapping given by $\sigma(x, y) = (g(x, y), f(x, y)) := (u, v)$ and such that $\sigma^{-1}(0, 0) = \{(0, 0)\}$. Then every local analytic curve $h(x, y) = 0$ has a well-defined *direct image* $\sigma^*(h = 0)$ which is an analytic curve in the target space (see [Ca2]). The Newton diagram of the direct image is characterized by two properties:

1. If h is an irreducible power series then $\Delta_{\sigma^*(h=0)} = \left\{ \frac{(f, h)_0}{(g, h)_0} \right\}$, where $(r, s)_0$ denotes the intersection multiplicity of the curves $r = 0$ and $s = 0$ at the origin.
2. If $h = h_1 h_2$ then $\Delta_{\sigma^*(h=0)} = \Delta_{\sigma^*(h_1=0)} + \Delta_{\sigma^*(h_2=0)}$.

Let $\text{jac}(g, f) = \frac{\partial g}{\partial x} \frac{\partial f}{\partial y} - \frac{\partial g}{\partial y} \frac{\partial f}{\partial x}$ be the jacobian determinant of the mapping σ . The direct image (see Preliminaries in [Ca2]) of $\text{jac}(g, f) = 0$ by σ is called the *discriminant curve*. We will write $\mathcal{N}_J(g, f)$ for the Newton diagram of the discriminant curve and following Teissier (see [Te3]) call it the *jacobian Newton diagram* of the morphism $\sigma = (g, f)$.

4 Approximate jacobian Newton diagrams of a branch

In this section we introduce the notion of the approximate jacobian Newton diagrams of an irreducible plane curve and we compute them. In what follows a branch $f(x, y) = 0$ will be given by an irreducible Weierstrass polynomial.

Let f be an irreducible Weierstrass polynomial and let $f^{(k)}$, for $0 \leq k \leq g - 1$, be the characteristic approximate roots of f . The jacobian Newton diagram $\mathcal{N}_J(f^{(k)}, f)$ is called the *k-th approximate jacobian Newton diagram of the branch $f(x, y) = 0$* .

The following result about the factorization of the jacobian $\text{jac}(f^{(k)}, f)$ is the main result of this note:

Theorem 1 *Let $f \in \mathbf{C}\{x\}[y]$ be an irreducible Weierstrass polynomial with Puiseux characteristic (b_0, \dots, b_g) . Let $f^{(k)}$, $0 \leq k \leq g-1$, be the k -th characteristic approximate root of f . Then the jacobian $\text{jac}(f^{(k)}, f)$ admits a factorization*

$$\text{jac}(f^{(k)}, f) = \Gamma^{(k+1)} \dots \Gamma^{(g)},$$

where the factors $\Gamma^{(i)}$ are not necessary irreducible, x is co-prime to the product $\Gamma^{(k+2)} \dots \Gamma^{(g)}$ and such that

1. If α is a Newton-Puiseux root of $\Gamma^{(k+1)}$ then $\text{cont}(\alpha, f) < b_{k+1}/b_0$.
2. If α is a Newton-Puiseux root of $\Gamma^{(i)}$, $k+2 \leq i \leq g$ then $\text{cont}(\alpha, f) = b_i/b_0$.
3. The intersection multiplicity $(\Gamma^{(i)}, x)_0 = n_1 \dots n_{i-1}(n_i - 1)$ for $k+2 \leq i \leq g$.

The proof of Theorem 1 will be done in Section 5.

The contacts between Newton-Puiseux roots of $\Gamma^{(k+1)}$ and f are not determined by the Puiseux characteristic of f as the following example shows.

Example 2 *Let $f = (y^3 - 6x^3y - x^4)^2 - 9x^9$ be the Weierstrass polynomial from Example 1 and let $g = (y^3 - x^4)^2 + x^9 - x^7y^2$. Both series f and g are irreducible with the same Puiseux characteristic $(6, 8, 11)$. The jacobian $\text{jac}(f^{(1)}, f) = 243x^8(y^2 - 2x^3)$ has two Newton-Puiseux roots $\alpha_1(x) = \sqrt{2}x^{3/2} + \dots$, $\alpha_2(x) = -\sqrt{2}x^{3/2} + \dots$ and $\text{cont}(\alpha_i, f) = \frac{4}{3} < \frac{b_2}{b_0}$ for $i = 1, 2$.*

On the other hand there are four Newton-Puiseux roots $\beta_1(x) = 0$, $\beta_2(x) = \frac{8}{27}x^2 + \dots$, $\beta_3(x) = \sqrt{\frac{21}{27}}x + \dots$, $\beta_4(x) = -\sqrt{\frac{21}{27}}x + \dots$ of $\text{jac}(g^{(1)}, g) = x^6y(21y^3 - 27x^2y + 8x^4)$ and $\text{cont}(\beta_i, g) = \frac{4}{3}$ for $i = 1, 2$, but $\text{cont}(\beta_i, g) = 1$ for $i = 3, 4$.

Further we will use the following property of the intersection multiplicity which is a consequence of the Noether's formula (see [G-P12] Proposition 3.3):

Property 1 *Let $g(x, y)$, $h(x, y)$ be irreducible power series co-prime to x . Then for fixed g , the function $h \mapsto \frac{(g, h)_0}{(x, h)_0}$ depends only on the contact $\text{cont}(g, h)$ and is a strictly increasing function of this quantity.*

Corollary 1 *Under assumptions and notations of Theorem 1 the jacobian Newton diagram of the mapping $(f^{(k)}, f)$ has the canonical decomposition*

$$\mathcal{N}_J(f^{(k)}, f) = \sum_{i=k+1}^g \left\{ \frac{(f, \Gamma^{(i)})_0}{(f^{(k)}, \Gamma^{(i)})_0} \right\}.$$

Proof. We prove that for every irreducible factor h of $\text{jac}(f^{(k)}, f)$ the quotient $\frac{(f, h)_0}{(f^{(k)}, h)_0}$ depends only on the contact $\text{cont}(f, h)$. Indeed there are two cases:

if $\text{cont}(f, h) < b_{k+1}/b_0$ then by the strong triangle inequality $\text{cont}(f^{(k)}, h) = \text{cont}(f, h)$ hence $\frac{(h, f^{(k)})_0}{(x, f^{(k)})_0} = \frac{(h, f)_0}{(x, f)_0}$ and we get

$$\frac{(f, h)_0}{(f^{(k)}, h)_0} = \frac{(x, f)_0}{(x, f^{(k)})_0}, \quad (1)$$

if $\text{cont}(f, h) > b_{k+1}/b_0$ then also by the strong triangle inequality $\text{cont}(f^{(k)}, h) = \text{cont}(f^{(k)}, f)$ hence $\frac{(f^{(k)}, h)_0}{(x, h)_0} = \frac{(f^{(k)}, f)_0}{(x, f)_0}$ and we get

$$\frac{(f, h)_0}{(f^{(k)}, h)_0} = \frac{(x, f)_0}{(f^{(k)}, f)_0} \cdot \frac{(f, h)_0}{(x, h)_0}. \quad (2)$$

Fix $i \in \{k+1, \dots, g\}$ and write $\Gamma^{(i)}$ as a product $h_1 \cdots h_r$ of irreducible factors h_j for $1 \leq j \leq r$. Then the Newton diagram of the direct image of the curve $\Gamma^{(i)} = 0$ is the sum $\sum_{j=1}^r \left\{ \frac{(f, h_j)_0}{(f^{(k)}, h_j)_0} \right\}$. Since all elementary Newton diagrams in the above sum have the same inclination one has

$$\sum_{j=1}^r \left\{ \frac{(f, h_j)_0}{(f^{(k)}, h_j)_0} \right\} = \left\{ \frac{\sum_{j=1}^r (f, h_j)_0}{\sum_{j=1}^r (f^{(k)}, h_j)_0} \right\} = \left\{ \frac{(f, \Gamma^{(i)})_0}{(f^{(k)}, \Gamma^{(i)})_0} \right\}.$$

We proved that the jacobian Newton diagram $\mathcal{N}_J(f^{(k)}, f)$ is the sum of elementary Newton diagrams from the statement of the Corollary. The inclination of the first elementary Newton diagram is given by formula (1) which can be written as $\frac{(x, f)_0}{(f^{(k)}, f)_0} \cdot \frac{(f, f^{(k)})_0}{(x, f^{(k)})_0}$. The inclinations of the remaining elementary Newton diagrams are given by formula (2). By Property 1 these inclinations form a strictly increasing sequence. This finishes the proof. ■

Now our aim is to give an arithmetical formula for $\mathcal{N}_J(f^{(k)}, f)$.

Put $\overline{b_k} := (f, f^{(k-1)})_0$ for $k \in \{0, 1, \dots, g\}$. Following Zariski (see [Z]), the set $\{\overline{b_0}, \overline{b_1}, \dots, \overline{b_g}\}$ is a minimal system of generators of the *semigroup*

$$\Gamma(f) := \{(f, g)_0 : f \text{ is not a factor of } g\}$$

of the branch $f(x, y) = 0$. This system of generators is uniquely determined by the Puiseux characteristic of f in the following way: $\overline{b_0} = b_0$, $\overline{b_1} = b_1$ and $\overline{b_q} = n_{q-1}\overline{b_{q-1}} + b_q - b_{q-1}$ for $2 \leq q \leq g$. Recall that $n_i = l_{i-1}/l_i$, where $l_i = \gcd(b_0, \dots, b_i) = \gcd(\overline{b_0}, \dots, \overline{b_i})$.

Remember that the *Milnor number* of a curve $g(x, y) = 0$ is by definition the intersection multiplicity $\left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y} \right)_0$.

Theorem 2 *Let $f = 0$, where f is an irreducible Weierstrass polynomial, be a branch with semigroup $\Gamma(f) = \langle \overline{b_0}, \dots, \overline{b_g} \rangle$. Then the canonical decomposition of the k -th approximate jacobian Newton diagram of f is*

$$\mathcal{N}_J(f^{(k)}, f) = \left\{ \frac{l_k(\mu(f^{(k)}) + \overline{m} - 1)}{\mu(f^{(k)}) + \overline{m} - 1} \right\} + \sum_{i=k+2}^g \left\{ \frac{(n_i - 1)\overline{b}_i}{\overline{m}n_{k+2} \cdots n_{i-1}(n_i - 1)} \right\},$$

where $\overline{m} = \overline{b_{k+1}}/l_{k+1}$, and $\mu(f^{(k)})$ is the Milnor number of $f^{(k)} = 0$.

Proof. In the course of the proof we shall use the canonical decomposition of $\mathcal{N}_J(f^{(k)}, f)$ from Corollary 1. We shall express all intersection multiplicities $(f, \Gamma^{(i)})_0$ and $(f^{(k)}, \Gamma^{(i)})_0$ for $k+1 \leq i \leq g$ in terms of the generators of the semigroup $\Gamma(f)$.

First consider $\Gamma^{(i)}$ for $k+2 \leq i \leq g$. By Theorem 1 the contact of every irreducible factor of $\Gamma^{(i)}$ with f equals b_i/b_0 . By Property 1 and Theorem 1:

$$(f, \Gamma^{(i)})_0 = (x, \Gamma^{(i)})_0 \frac{(f, \Gamma^{(i)})_0}{(x, \Gamma^{(i)})_0} = (x, \Gamma^{(i)})_0 \frac{(f, f^{(i-1)})_0}{(x, f^{(i-1)})_0} = (n_i - 1)\overline{b}_i. \quad (3)$$

By Corollary 1 and equality (2)

$$\frac{(f, \Gamma^{(i)})_0}{(f^{(k)}, \Gamma^{(i)})_0} = \frac{(f, f^{(i-1)})_0}{(f^{(k)}, f^{(i-1)})_0} = \frac{(x, f)_0}{(f^{(k)}, f)_0} \cdot \frac{(f, f^{(i-1)})_0}{(x, f^{(i-1)})_0} = \frac{l_{i-1}\overline{b}_i}{\overline{b}_{k+1}}.$$

Hence by (3)

$$(f^{(k)}, \Gamma^{(i)})_0 = \frac{\overline{b_{k+1}}}{l_{i-1}\overline{b}_i} (f, \Gamma^{(i)})_0 = \overline{m}n_{k+2} \cdots n_{i-1}(n_i - 1).$$

In order to compute $(f^{(k)}, \Gamma^{(k+1)})_0$ we use Theorem 3.2 of [Ca1]. We get

$$(f^{(k)}, \text{jac}(f^{(k)}, f))_0 = \mu(f^{(k)}) + (f^{(k)}, f)_0 - 1.$$

Since $(f^{(k)}, \text{jac}(f^{(k)}, f))_0 = \sum_{i=k+1}^g (f^{(k)}, \Gamma^{(i)})_0$ we have

$$\begin{aligned} (f^{(k)}, \Gamma^{(k+1)})_0 &= \mu(f^{(k)}) + (f^{(k)}, f)_0 - 1 - \sum_{i=k+2}^g \overline{m}n_{k+2} \cdots n_{i-1}(n_i - 1) \\ &= \mu(f^{(k)}) + \overline{b_{k+1}} - 1 - \overline{m}(l_{k+1} - 1) = \mu(f^{(k)}) + \overline{m} - 1. \end{aligned}$$

Finally by Corollary 1 and equality (1)

$$\frac{(f, \Gamma^{(k+1)})_0}{(f^{(k)}, \Gamma^{(k+1)})_0} = \frac{(x, f)_0}{(x, f^{(k)})_0} = l_k$$

Hence $(f, \Gamma^{(k+1)})_0 = l_k(\mu(f^{(k)}) + \overline{m} - 1)$. ■

Remark 1 In the above proof we compute the inclinations of elementary Newton diagrams of the canonical decomposition of $\mathcal{N}_J(f^{(k)}, f)$ which are equal to $\frac{l_{i-1}\overline{b}_i}{\overline{b}_{k+1}}$ for $i \in \{k+1, \dots, g\}$. These inclinations are called jacobian invariants.

Example 3 Let $f(x, y) = (y^2 - x^3)^2 - x^5y$. Then $f = 0$ is a branch and $\Gamma(f) = \langle 4, 6, 13 \rangle$. The characteristic approximate roots of f are $f^{(0)} = y$ and $f^{(1)} = y^2 - x^3$. The factorization of $\text{jac}(f^{(0)}, f)$ described in Theorem 1 is $\text{jac}(f^{(0)}, f) = \Gamma^{(1)}\Gamma^{(2)}$, where $\Gamma^{(1)} = x^2$ and $\Gamma^{(2)} = 6y^2 + 5x^2y - 6x^3$. We also have $\text{jac}(f^{(1)}, f) = x^4(10y^2 + 3x^3)$. Finally $\mathcal{N}_J(f^{(0)}, f) = \{\frac{8}{2}\} + \{\frac{13}{3}\}$ and $\mathcal{N}_J(f^{(1)}, f) = \{\frac{28}{14}\}$.

Corollary 2 The family of the approximate jacobian Newton diagrams of a branch only depends on its topological type.

If f is an irreducible Weierstrass polynomial then $f^{(0)} = 0$ is a smooth curve. By Smith-Merle-Ephraim (see for example Theorem 2.2 of [GB-G2]) the approximate jacobian Newton diagram $\mathcal{N}_J(f^{(0)}, f)$ determines the topological type of the branch $f = 0$. Nevertheless we can also obtain the generators of the semigroup of the branch $f = 0$ using the whole family of its approximate jacobian Newton diagrams in an easy way: let $\Gamma(f) = \langle \bar{b}_0, \dots, \bar{b}_g \rangle$ be the semigroup of $f = 0$. It is clear that \bar{b}_0 is the smallest inclination of $\mathcal{N}_J(f^{(0)}, f)$. Denote by ι the inclination of the elementary diagram $\mathcal{N}_J(f^{(g-1)}, f)$. Put \mathcal{H}_r , for $r \in \{0, \dots, g-2\}$, the height of the last elementary diagram of $\mathcal{N}_J(f^{(r)}, f)$, that is the height of the elementary diagram of $\mathcal{N}_J(f^{(r)}, f)$ which has the biggest inclination. Then $\bar{b}_{r+1} = \frac{\iota \mathcal{H}_r}{\iota - 1}$ for $r \in \{0, \dots, g-2\}$. Finally $\bar{b}_g = \frac{\mathcal{L}}{\iota - 1}$, where \mathcal{L} is the length of the last elementary diagram of $\mathcal{N}_J(f^{(g-2)}, f)$.

Example 4 Consider the branches $f_i = 0$ for $i \in \{1, \dots, 4\}$ with semigroups $\Gamma(f_1) = \langle 4, 14, 31 \rangle$, $\Gamma(f_2) = \langle 4, 6, 35 \rangle$, $\Gamma(f_3) = \langle 4, 6, 37 \rangle$ and $\Gamma(f_4) = \langle 6, 10, 31 \rangle$. By Theorem 2 we have $\mathcal{N}_J(f_1^{(1)}, f_1) = \mathcal{N}_J(f_2^{(1)}, f_2) = \{\frac{72}{36}\}$ and $\mathcal{N}_J(f_3^{(1)}, f_3) = \mathcal{N}_J(f_4^{(1)}, f_4) = \{\frac{76}{38}\}$.

Given a branch $f = 0$, put \mathcal{F} its family of approximate jacobian Newton diagrams but the first one. The example shows that \mathcal{F} is not a complete topological invariant of a branch. The curves $f_3 = 0$ and $f_4 = 0$ have the same \mathcal{F} but they have different multiplicities at the origin. The curves $f_1 = 0$ and $f_2 = 0$ have the same \mathcal{F} and the same multiplicity at the origin but in spite of it they have different topological type.

5 Proof of Theorem 1

Let τ be a positive rational number and let $g(x, y) = \sum_{i \in \mathbf{Q}, j \in \mathbf{N}} a_{ij} x^i y^j \in \mathbf{C}\{x\}^*[y]$.

Put $w(x) := 1$ and $w(y) := \tau$ the *weights* of the variables x and y . By definition the *weighted order* of g is $\text{ord}_\tau(g) = \min\{i + \tau j : a_{ij} \neq 0\}$ and the *weighted initial part* of g is $\text{in}_\tau(g) = \sum_{i + \tau j = \text{ord}_\tau(g)} a_{ij} x^i y^j$.

Lemma 1 Let $g(x, y) = u(x, y) \cdot x^N \prod_{i=1}^d (y - \alpha_i(x))$, where $u(0, 0) \neq 0$, $N \in \mathbf{Q}$, $\alpha_i(x) = c_i x^\tau + \dots$ for $1 \leq i \leq k$ and $\text{ord}_x(\alpha_i(x)) < \tau$, for $k+1 \leq i \leq d$. Then $\text{in}_\tau(g) = c x^M \prod_{i=1}^k (y - c_i x^\tau)$ for some $c \in \mathbf{C}$ and some $M \in \mathbf{Q}$.

Proof. Observe that $\text{in}_\tau(y - \alpha_i(x)) = y - c_i x^\tau$ for $1 \leq i \leq k$ and $\text{in}_\tau(y - \alpha_i(x)) = -\text{in}_\tau \alpha_i(x)$ for $k+1 \leq i \leq d$. Since the initial part of a product is the product of the initial parts of every factor we get the lemma. ■

Lemma 2 Let $h_1, h_2 \in \mathbf{C}\{x\}^*[y]$ and $\tau \in \mathbf{Q}^+$. Assume that the jacobian $\text{jac}(\text{in}_\tau(h_1), \text{in}_\tau(h_2)) \neq 0$. Then $\text{in}_\tau(\text{jac}(h_1, h_2)) = \text{jac}(\text{in}_\tau(h_1), \text{in}_\tau(h_2))$.

Proof. For all monomials $M_1 = x^{i_1} y^{j_1}$ and $M_2 = x^{i_2} y^{j_2}$ we have $\text{jac}(M_1, M_2) = (i_1 j_2 - i_2 j_1) x^{i_1 + i_2 - 1} y^{j_1 + j_2 - 1}$ hence $\text{ord}_\tau(\text{jac}(M_1, M_2)) = \text{ord}_\tau(M_1) + \text{ord}_\tau(M_2) - 1 - \tau$ provided $i_1 j_2 - i_2 j_1 \neq 0$. It follows that $\text{jac}(\text{in}_\tau(h_1), \text{in}_\tau(h_2))$ is the sum of monomials of the same weighted order $\text{ord}_\tau(\text{in}_\tau(h_1)) + \text{ord}_\tau(\text{in}_\tau(h_2)) - 1 - \tau$ (that is a quasi-homogeneous polynomial). Moreover $\text{jac}(h_1, h_2) = \text{jac}(\text{in}_\tau(h_1) + (h_1 - \text{in}_\tau(h_1)), \text{in}_\tau(h_2) + (h_2 - \text{in}_\tau(h_2))) = \text{jac}(\text{in}_\tau(h_1), \text{in}_\tau(h_2)) + \text{terms of higher weighted order}$ which proves the lemma. ■

Recall that Newton-Puiseux roots of an irreducible Weierstrass polynomial $f \in \mathbf{C}\{x\}[y]$, $\deg f = n$ form a cycle: if $\gamma(x) = \sum a_i x^{i/n}$ is a root of f then other roots of f are $\gamma_j(x) = \sum a_i \omega_j^i x^{i/n}$, where ω_j is a n -th root of unity. Moreover $\text{ord}_x(\gamma(x) - \gamma_j(x)) \geq \frac{b_{k+1}}{b_0}$ if and only if ω_j is a l_k -th root of unity (see [Z]).

Let $f = \prod_{i=1}^n (y - \gamma_i(x))$ be an irreducible Weierstrass polynomial with Puiseux characteristic (b_0, \dots, b_g) and let $f^{(k)}(x, y) = \prod_{j=1}^m (y - \delta_j(x))$, where $n = m l_k$, be the characteristic approximate root of f . Put $J(x, y) := \text{jac}(f^{(k)}, f) = \text{unity} \cdot x^\alpha \prod_l (y - \sigma_l(x))$. In order to prove Theorem 1 we need

Lemma 3 Fix $\gamma \in \text{Zer} f$ and $\tau \in \mathbf{Q}$ such that $\tau \geq \frac{b_{k+1}}{b_0}$. Then

1. if $\frac{b_i}{b_0} < \tau \leq \frac{b_{i+1}}{b_0}$, where $j \in \{k+1, \dots, g\}$ then $\#\{i : \mathcal{O}(\sigma_i, \gamma) \geq \tau\} = l_j - 1$,
2. if $\tau = \frac{b_{k+1}}{b_0}$ then $\#\{i : \mathcal{O}(\sigma_i, \gamma) \geq \tau\} = n_{k+1}(l_{k+1} - 1)$.

Proof. Let $\tilde{J}(x, y) := J(x, y + \gamma(x))$, $\tilde{f}(x, y) := f(x, y + \gamma(x))$ and $\tilde{f}^{(k)}(x, y) := f^{(k)}(x, y + \gamma(x))$. Clearly $\tilde{J}(x, y) = \text{unity} \cdot x^\alpha \prod_l (y - (\sigma_l(x) - \gamma(x)))$. By Lemma 1 $\#\{i : \mathcal{O}(\sigma_i, \gamma) \geq \tau\} = \deg_y(\text{in}_\tau(\tilde{J}(x, y)))$.

Assume first that $\tau > \frac{b_{k+1}}{b_0}$ and $\tau \neq \frac{b_i}{b_0}$ for all $j \in \{k+2, \dots, g\}$. The weighted initial part of $\tilde{f}(x, y) = \prod_{i=1}^n (y - (\gamma_i(x) - \gamma(x)))$ is equal to $\text{in}_\tau(\tilde{f}(x, y)) = c_1 x^{\alpha_1} y^{d(\tau)}$, where $c_1 \in \mathbf{C} \setminus \{0\}$ and $d(\tau) := \#\{i : \mathcal{O}(\gamma_i, \gamma) \geq \tau\}$. More precisely if $\frac{b_i}{b_0} < \tau < \frac{b_{i+1}}{b_0}$ then $d(\tau) = l_j$.

Consider the function $\tilde{f}^{(k)}(x, y) = \prod_{j=1}^m (y - (\delta_j(x) - \gamma(x)))$. Since $\mathcal{O}(\delta_j, \gamma) < \tau$ for every $j \in \{1, \dots, m\}$, we get by Lemma 1 $\text{in}_\tau \tilde{f}^{(k)}(x, y) = c_2 x^{\alpha_2}$, where $c_2 \in \mathbf{C} \setminus \{0\}$.

Using Lemma 2 we get

$$\text{in}_\tau(\tilde{J}(x, y)) = \text{jac}(c_2 x^{\alpha_2}, c_1 x^{\alpha_1} y^{d(\tau)}) = c_1 c_2 \alpha_2 d(\tau) x^{\alpha_1 + \alpha_2 - 1} y^{d(\tau) - 1},$$

so its y -degree is equal to $d(\tau) - 1 = l_j - 1$ for $\frac{b_j}{b_0} < \tau < \frac{b_{j+1}}{b_0}$.

Let us choose $\tau < \frac{b_{j+1}}{b_0}$ close enough to $\frac{b_{j+1}}{b_0}$ that no σ_i satisfies $\tau \leq \mathcal{O}(\sigma_i, \gamma) < \frac{b_{j+1}}{b_0}$. Then $\#\{i : \mathcal{O}(\sigma_i, \gamma) \geq \tau\} = \#\{i : \mathcal{O}(\sigma_i, \gamma) \geq \frac{b_{j+1}}{b_0}\}$ and the proof of statement 1 is done.

Assume now that $\tau = \frac{b_{k+1}}{b_0}$. By Lemma 1

$$\begin{aligned} \text{in}_\tau \tilde{f}(x, y) &= x^{\alpha_3} \prod_{\omega^{l_k}=1} (y - a(\omega^{b_{k+1}} - 1)x^{b_{k+1}/b_0}) \\ &= x^{\alpha_3} \prod_{\omega^{l_k}=1} \left[(y + ax^{b_{k+1}/b_0}) - a\omega^{b_{k+1}} x^{b_{k+1}/b_0} \right] \\ &= x^{\alpha_3} \left[(y + ax^{b_{k+1}/b_0})^{n_{k+1}} - (ax^{b_{k+1}/b_0})^{n_{k+1}} \right]^{l_{k+1}}, \end{aligned}$$

where $\omega \in \mathbf{C}$ and a is the coefficient in γ of the term x^{b_{k+1}/b_0} . The last equality follows from the formula $\prod_{\omega^p=1} (Z - b\omega^q) = \left(Z^{\frac{p}{\gcd(p,q)}} - b^{\frac{p}{\gcd(p,q)}} \right)^{\gcd(p,q)}$.

Moreover and also using Lemma 1 we have $\text{in}_\tau \tilde{f}^{(k)}(x, y) = x^{\alpha_4} (y + ax^{b_{k+1}/b_0})$ since there is only one Newton-Puiseux root δ_j of $f^{(k)}$ such that $\mathcal{O}(\delta_j, \gamma) \geq \frac{b_{k+1}}{b_0}$ (otherwise if there were two of such roots $\delta_{j_1}, \delta_{j_2}$ then by the triangular property of the contact order we obtain $\mathcal{O}(\delta_{j_1}, \delta_{j_2}) \geq \frac{b_{k+1}}{b_0}$ which is not possible).

We prove now the equality $\alpha_3 = \alpha_4 l_k$. Note that $\alpha_3 = \sum_{i \in I'} \mathcal{O}(\gamma_i, \gamma)$ and $\alpha_4 = \sum_{j \in J'} \mathcal{O}(\delta_j, \gamma)$, where $I' := \{i : \mathcal{O}(\gamma_i, \gamma) < \frac{b_{k+1}}{b_0}\}$ and $J' := \{j : \mathcal{O}(\delta_j, \gamma) < \frac{b_{k+1}}{b_0}\}$. Using Puiseux characteristic of f and after Section 3 in [G-Pl3] we obtain $\alpha_3 = \sum_{i \in I'} \mathcal{O}(\gamma_i, \gamma) = \sum_{l=1}^k \#\{i : \mathcal{O}(\gamma_i, \gamma) = \frac{b_l}{b_0}\} \cdot \frac{b_l}{b_0} = (n - l_1) \frac{b_1}{b_0} + \dots + (l_{k-1} - l_k) \frac{b_k}{b_0}$ and by the same argument $\alpha_4 = \sum_{j \in J'} \mathcal{O}(\delta_j, \gamma) = \left(\frac{n}{l_k} - \frac{l_1}{l_k} \right) \frac{b_1}{b_0} + \dots + \left(\frac{l_{k-1}}{l_k} - 1 \right) \frac{b_k}{b_0}$.

Finally the initial part of \tilde{J} is

$$\text{in}_\tau(\tilde{J}) = \text{jac}(\text{in}_\tau(\tilde{f}^{(k)}), \text{in}_\tau(\tilde{f})) = \text{jac}(v, (v^{n_{k+1}} - a^{n_{k+1}} u^\theta)^{l_{k+1}}),$$

where $v = x^{\alpha_4} (y + ax^{b_{k+1}/b_0})$, $u = x$ and $\theta = n_{k+1} \left(\frac{b_{k+1}}{b_0} + \alpha_4 \right)$ so $\text{in}_\tau(\tilde{J}) = \frac{\partial \text{in}_\tau(\tilde{f})}{\partial u} \frac{\partial v}{\partial y}$ and its y -degree is equal to $n_{k+1}(l_{k+1} - 1)$. ■

Remark 2 The proof of Merle formula in [G-Pl1] was based on the equality $\Delta_{\tilde{f}} = \Delta_{\tilde{j}} + \{\frac{\infty}{1}\}$, where $\tilde{j}(x, y) = j(x, y + \gamma(x))$ and $j(x, y) := \text{jac}(x, f)$. Note that the statement of Lemma 3 can be written as $\deg_y \text{in}_\tau(\tilde{J}(x, y)) =$

$\deg_y \text{in}_\tau(\tilde{f}(x, y)) - 1$ for $\tau > \frac{b_{k+1}}{b_0}$. It follows from this equality that $\tilde{\Delta}_{\tilde{f}} = \tilde{\Delta}_{\tilde{f}} + \{\frac{\infty}{1}\}$, where $\tilde{\Delta}_{\tilde{f}}$ and $\tilde{\Delta}_{\tilde{f}}$ are the sums of elementary Newton diagrams in the canonical decompositions of $\Delta_{\tilde{f}}$ and $\Delta_{\tilde{f}}$ respectively with inclinations bigger than $\frac{b_{k+1}}{b_0}$.

Corollary 3 *Keep the above notations and put $\tau_i := \text{cont}(\sigma_i, f)$. Then*

1. *if $\tau_i \geq \frac{b_{k+1}}{b_0}$ then $\tau_i \in \left\{ \frac{b_{k+2}}{b_0}, \dots, \frac{b_g}{b_0} \right\}$.*
2. *The number $\#\{i : \tau_i = \frac{b_j}{b_0}\} = n_1 \cdots n_{j-1}(n_j - 1)$ for $j \in \{k+2, \dots, g\}$.*

Proof. First take τ such that $\frac{b_j}{b_0} < \tau \leq \frac{b_{j+1}}{b_0}$ for $k+1 \leq j \leq g$. We shall prove that

$$\#\{i : \tau_i \geq \tau\} = n - n_1 \cdots n_j. \quad (4)$$

In the set $\text{Zer}f$ we define the equivalence relation given by

$$\gamma^* \equiv \gamma' \text{ if and only if } \mathcal{O}(\gamma^*, \gamma') \geq \frac{b_{j+1}}{b_0}.$$

Put $I_\gamma := \{i : \mathcal{O}(\sigma_i, \gamma) \geq \tau\}$ for $\gamma \in \text{Zer}f$. By Lemma 3 we get $\#I_\gamma = l_j - 1$. Note that $I_{\gamma'} = I_{\gamma^*}$ for $\gamma^* \equiv \gamma'$ and $I_{\gamma'} \cap I_{\gamma^*} = \emptyset$ when $\gamma^* \not\equiv \gamma'$.

Remark that $n_1 \cdots n_j$ is the number of cosets in the equivalence relation \equiv . Since $\#\{i : \tau_i \geq \tau\} = \bigcup_{\gamma \in \text{Zer}f} I_\gamma$ we have $\#\{i : \tau_i \geq \tau\} = n_1 \cdots n_j \cdot \#I_\gamma = n_1 \cdots n_j(l_j - 1) = n - n_1 \cdots n_j$. The equality (4) is proved.

Fix small positive number ϵ such that

$$\#\{i : \tau_i = \tau\} = \#\{i : \tau_i \geq \tau\} - \#\{i : \tau_i \geq \tau + \epsilon\}.$$

If $\tau \neq \frac{b_j}{b_0}$ for all $j \in \{k+2, \dots, g\}$ the above difference is equal to zero. If $\tau = \frac{b_j}{b_0}$ for some $j \in \{k+2, \dots, g\}$, then $\#\left\{i : \tau_i = \frac{b_j}{b_0}\right\} = (n - n_1 \cdots n_{j-1}) - (n - n_1 \cdots n_j) = n_1 \cdots n_{j-1}(n_j - 1)$.

Finally using the same argument as before (for $\tau = \frac{b_{k+1}}{b_0}$) we have

$$\begin{aligned} \#\left\{i : \tau_i = \frac{b_{k+1}}{b_0}\right\} &= \#\left\{i : \tau_i \geq \frac{b_{k+1}}{b_0}\right\} - \#\left\{i : \tau_i \geq \frac{b_{k+1}}{b_0} + \epsilon\right\} \\ &= \#\left\{i : \tau_i \geq \frac{b_{k+1}}{b_0}\right\} - (n - n_1 \cdots n_{k+2}) \\ &= n_{k+1}(l_{k+1} - 1)n_1 \cdots n_k - (n - n_1 \cdots n_{k+1}) = 0. \end{aligned}$$

■

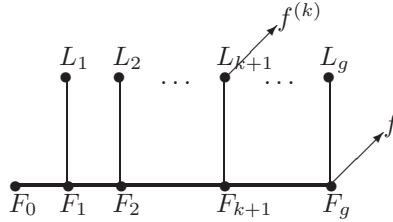
Proof of Theorem 1.- Let $k+2 \leq j \leq g$. Put $\Gamma^{(j)} = \prod (y - \sigma_i(x))$, where the product runs over σ_i with $\text{cont}(\sigma_i, f) = \frac{b_j}{b_0}$ and let $\Gamma^{(k+1)} = \frac{\text{jac}(f^{(k)}, f)}{\Gamma^{(k+2)} \dots \Gamma^{(g)}}$. It follows from the first statement of Corollary 3 that for every Newton-Puiseux root $\alpha \in \text{Zer}\Gamma^{(k+1)}$ we have $\text{cont}(\alpha, f) < \frac{b_{k+1}}{b_0}$. Finally by the second statement of Corollary 3 we get $(\Gamma^{(i)}, x)_0 = n_1 \dots n_{i-1}(n_i - 1)$ for $k+2 \leq i \leq g$.

6 Relation with Michel's theorem

In [Mi] the author considered a finite morphism $(f, g) : (X, p) \longrightarrow (\mathbb{C}^2, 0)$, where (X, p) is a normal germ of complex surface. Michel determined the jacobian quotients via a good minimal resolution and pointed out the importance of the multiplicities of the jacobian quotients. More precisely and following notation of [Mi], let R be a good resolution of (f, g) and put $E = R^{-1}(p)$ the exceptional divisor of R . For every irreducible component E_i of E , denote E'_i the set of points of E_i which are smooth points of the total transform $\tilde{E} = R^{-1}((fg)^{-1}(0))$. Denote the order of $f \circ R$ (respectively $g \circ R$) at a generic point of E_i $v(f, E_i)$ (respectively $v(g, E_i)$). The quotient $q_i = \frac{v(g, E_i)}{v(f, E_i)}$ is the *Hironaka number* of E_i .

Let q be a Hironaka number and put $E(q)$ the union of the E'_i such that $q_i = q$ to which we add $E_i \cap E_j$ if $q_i = q_j = q$. Let $\{E^k(q)\}_k$ be the connected components of $E(q)$. By definition a q -zone is a connected component of $E(q)$ and a q -zone is a *rupture zone* if there exists in it at least one E'_i with negative Euler characteristic. Then after Theorem 4.8 of [Mi] the set of jacobian invariants of the morphism (f, g) is equal to the set of Hironaka numbers q such that there exists at least one q -zone in E which is a rupture zone.

Consider an irreducible Weierstrass polynomial f with Puiseux characteristic (b_0, b_1, \dots, b_g) , where $b_0 < b_1$ (i.e. $x = 0$ is transverse to $f = 0$). Below is the schematic picture of the resolution graph of the curve $f^{(k)}f = 0$.



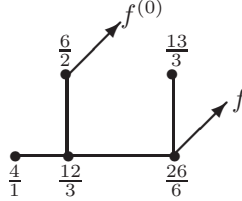
Every jacobian invariant $q \in \left\{ l_k, \frac{l_{k+1}\overline{b_{k+2}}}{b_{k+1}}, \dots, \frac{l_{g-1}\overline{b_g}}{b_{k+1}} \right\}$ of the morphism $(f^{(k)}, f)$ corresponds to exactly one rupture zone.

The rupture zone for $q = l_k$ is the tree with endpoints $F_0, F_{k+1}, L_1, \dots, L_k$. It yields the factor $\Gamma^{(k+1)}$ of the jacobian and by Michel's theorem $(\Gamma^{(k+1)}, h)_0 = \sum_{i=1}^{k+1} v(h, F_i) - \sum_{i=1}^k v(h, L_i) - v(h, F_0)$, where $h = f$ or $h = f^{(k)}$.

Every rupture zone for $q = \frac{l_{i-1}\overline{b_i}}{b_{k+1}}$, where $k+2 \leq i \leq g$ is the bamboo with endpoints F_i and L_i . It yields the factor $\Gamma^{(i)}$ of the jacobian and by Michel's

theorem $(\Gamma^{(i)}, h)_0 = v(h, F_i) - v(h, L_i)$ for $k + 2 \leq i \leq g$, where $h = f$ or $h = f^{(k)}$.

As an illustration we draw the resolution graph of $f^{(0)}f = 0$, where f is the Weierstrass polynomial from Example 3. The labels of divisors are Hironaka numbers written in the form $\frac{v(f, E_i)}{v(f^{(0)}, E_i)}$.



There are two rupture zones corresponding to Hironaka numbers 4 and $\frac{13}{3}$. It follows from [Mi] that $\mathcal{N}_J(f^{(0)}, f) = \{\frac{12}{3}\} - \{\frac{4}{1}\} + \{\frac{26}{6}\} - \{\frac{13}{3}\} = \{\frac{8}{2}\} + \{\frac{13}{3}\}$.

Remark 3 Remark that Theorem 1 is also true when we change $f^{(k)}$ for any irreducible Weierstrass polynomial with the properties of statement of Proposition 2.

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